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Surfing on solitary waves

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The steady motion of a flat surfboard propelled by a solitary wave is considered. The shape of the free surface and the flow of the fluid are determined numerically by series truncation for flows without spray or splash. These flows all bifurcate from the uniform horizontal flow at the critical value of the Froude number. Various limiting cases of these special flows are described analytically. Flows past submerged hydrofoils are discussed also.

1. Introduction

Surfing is the motion of a relatively flat object propelled by a wave along the surface of a fluid. The object alters the motion and the surface of the fluid, while the fluid exerts forces and moments on the object. We shall determine the fluid motion, the surface shape, and the forces on a flat surfboard on a two-dimensional potential flow when the unaltered fluid motion is a solitary wave.

The horizontal force on a surfboard in an inviscid fluid is due to wave-making and to spray or splash. In two dimensions a surfboard carried by a solitary wave does not make waves because the flow is supercritical. We shall show that there are particular flows without spray or splash either. Thus for these special flows, there is no net horizontal force exerted on the surfboard, while for flows with a splash jet there is a net horizontal drag force.

In addition to the dynamic force, there is a buoyancy force acting on the board. It balances the normal component of the weight of the board and the surfer. The tangential component of the weight is balanced by the viscous force. Therefore we shall omit these forces from our analysis.

First, we shall consider a horizontal board riding on top of a wave, without a spray or splash jet, as is shown in figure 1. We shall find a two-parameter family of such flows. The parameters are L/H and W/H where L and W are the board length and height, and H is the water depth at infinity. These flows all bifurcate from the uniform horizontal flow, for which $W/H = 1$, at the critical value $F = 1$ of the Froude number.

Next we shall consider a sloping board riding on the face of a wave. For such a board, we shall calculate a two-parameter family of flows without jets for which there is no force on the board. These flows also bifurcate from the uniform stream at $F = 1$.

Finally we shall indicate what the corresponding results would be for a submerged flat hydrofoil.

These surfing flows can be compared with planing flows on deep water. In planing, the free surface without the planing object is flat. Furthermore, waves and wave drag always occur in deep water planing, and usually splash and splash drag occurs as

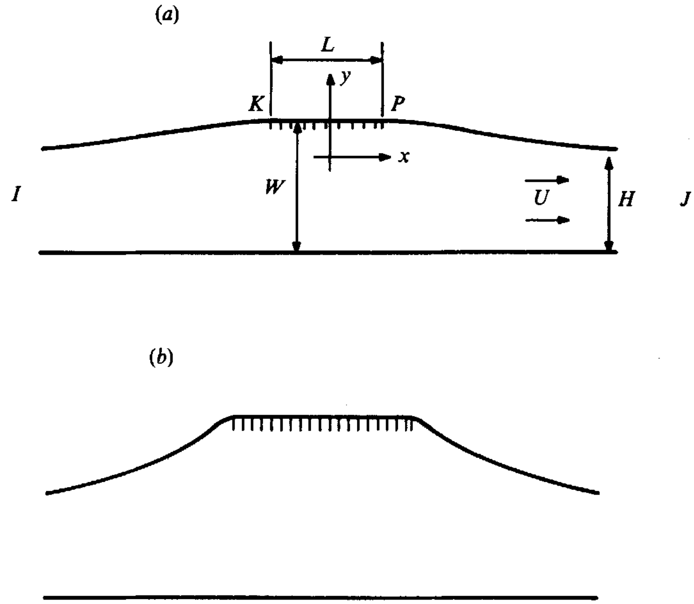

FIGURE 1. (a) The flow past a horizontal flat plate of length L at height W above the bottom. The endpoints of the plate are at K and P. The flow has depth H and velocity U at $x = \pm\infty$. These points are denoted I and J. The profile shown is a computed solution for $F^2 = 1.4$ and $\gamma = 5\pi/16$. The horizontal and vertical scales are equal, and the coordinate axes are shown. (b) Computed solution for flow past a horizontal plate with $F^2 = 1.9$ and $\gamma = 5\pi/16$.

well. Therefore an external propulsive force is required to maintain the motion. Such flows have been calculated by Green (1936), neglecting gravity, and by Rispin (1967), Wu (1967), Dagan & Tulin (1972), Ting & Keller (1974), and Naghdi & Rubin (1981), including gravity. Other authors have calculated them by using linear free-surface theory (Wagner 1932; Cumberbatch 1958). Cumberbatch (1958) and Ting & Keller (1977) also showed that the splash jet can be eliminated by shaping the planing object properly.

In §2 we formulate the problem for a horizontal board and present the numerical method for solving it. In §3 we give the numerical results and describe various limiting cases. In §4 we treat the sloping board. In §5 we discuss flow past a submerged hydrofoil and in §6 we state the conclusions.

2. Formulation

Let us consider the flow in the region shown in figure 1. The horizontal bottom IJ is a streamline on which we require $\psi = 0$. We choose Cartesian coordinates with the x-axis parallel to the bottom and the y-axis directed vertically upwards. Gravity g is acting in the negative y-direction. As $|x| \to \infty$, the flow is required to approach a uniform stream with constant velocity U in the x-direction and uniform depth H. The coordinate y is measured from the level of the free surfaces at $|x| = \infty$. The free surfaces IK and PJ and the horizontal plate KP are part of a streamline on which $\psi = UH$. We denote by L the length of the plate KP.

Let the complex potential be $f = \phi + \mathrm{i}\psi$. Without loss of generality we choose

$\phi = 0$ at the middle of the plate. On the free surfaces, where the pressure is constant, the Bernoulli equation yields

$$\tfrac{1}{2}q^2 + gy = \tfrac{1}{2}U^2 \quad \text{on } IK \text{ and } PJ. \tag{2.1}$$

Hence q denotes the magnitude of the velocity.

We choose H as the unit of length and U as the unit of velocity. Then (2.1) becomes

$$q^2 + \frac{2}{F^2}y = 1 \quad \text{on } IK \text{ and } PJ. \tag{2.2}$$

Here F is the Froude number defined by

$$F = U/(gH)^{\frac{1}{2}}. \tag{2.3}$$

Now we consider the plane of the complex potential $f = \phi + \mathrm{i}\psi$. In it let $\pm b$ denote the values of the potential function at the two separation points, P and K on $\psi = 1$.

Let the complex velocity be $\zeta = u - \mathrm{i}v$, where u and v are the x- and y-components of the velocity. As $\phi \to \pm\infty$, the flow approaches a uniform stream with constant unit velocity. For $F > 1$, we expect the approach to be described by exponentially decreasing terms, so we write

$$\zeta \sim 1 + D\,\mathrm{e}^{\mp\pi\lambda f} \quad \text{as } \phi \to \pm\infty. \tag{2.4}$$

Here D is a constant to be found as part of the solution and λ is the smallest positive root of

$$F^2\pi\lambda - \tan\pi\lambda = 0. \tag{2.5}$$

By analogy with other free-streamline problems, we expect ζ to behave at the edges like

$$\zeta \sim G + H(f \pm b)^{\frac{1}{2}} \quad \text{as } f \to \mp b. \tag{2.6}$$

Here G and H are constants to be found as part of the solution.

The problem is to find ζ as an analytic function of $f = \phi + \mathrm{i}\psi$ in the strip $0 < \psi < 1$, satisfying (2.2), (2.4), (2.6) and the kinemetic condition

$$v = 0 \quad \text{on } KP \quad \text{and } IJ. \tag{2.7}$$

We define the new variable t by the relation

$$f = \frac{2}{\pi}\log\frac{1+t}{1-t}. \tag{2.8}$$

The transformation (2.8) maps the flow domain into the interior of the unit circle in the t-plane. The bottom IJ goes onto the real diameter and the free surface and the plate KP go onto that portion of the circumference lying in the upper half of the t-plane. The points $t = \mathrm{e}^{\mathrm{i}\gamma}$ and $t = -\mathrm{e}^{-\mathrm{i}\gamma}$ are the images in the complex t-plane of the separation points P and K. By using (2.8) we find that b and γ are related by

$$\gamma = 2\tan^{-1}\mathrm{e}^{-\pi b/2}. \tag{2.9}$$

We use the notation $t = r\mathrm{e}^{\mathrm{i}\sigma}$ so that the free surfaces and the plate are described by $r = 1$ and $0 \leqslant \sigma \leqslant \pi$.

We represent the complex velocity ζ by

$$\zeta = \mathrm{e}^{\Omega(t)}, \tag{2.10}$$

where $\Omega(t)$ has the expansion

$$\Omega(t) = A(1-t^2)^{2\lambda} + B[(t^2+1)^2 - 4t^2\cos^2\gamma]^{\frac{1}{2}} - B(4-4\cos^2\gamma)^{\frac{1}{2}} + \sum_{n=1}^{\infty} a_n(t^{2n}-1). \tag{2.11}$$

It can be checked that the expression (2.10) satisfies (2.4) and (2.6). The kinematic condition (2.7) on IJ is automatically satisfied by requiring the coefficient a_n to be real. For given values of F and γ, the coefficient a_n and the constants A and B have to be determined to make (2.10) satisfy the Bernoulli condition (2.2) and the kinematic condition (2.7) on the plate KP. Once A, B, and the a_n are known, the length L of the plate and the free-surface profiles can be obtained by integrating numerically the identity

$$\frac{\partial x}{\partial \phi} + \mathrm{i}\frac{\partial y}{\partial \phi} = \frac{1}{\zeta}. \tag{2.12}$$

We shall solve the problem approximately by truncating the infinite sum in (2.11) after N terms. Differentiating (2.2) with respect to σ and using (2.8) yields

$$F^2[\tilde{u}(\sigma)\,\tilde{u}_\sigma(\sigma) + \tilde{v}(\sigma)\,\tilde{v}_\sigma(\sigma)] - \frac{2}{\pi}\frac{\tilde{v}(\sigma)}{[\tilde{u}(\sigma)]^2 + [\tilde{v}(\sigma)]^2}\frac{1}{\sin\sigma} = 0. \tag{2.13}$$

Here $\tilde{u}(\sigma)$ and $\tilde{v}(\sigma)$ are the components of the velocity on the free surfaces and on the plate.

We now introduce the $N+2$ mesh points,

$$\sigma_1 = \frac{\pi}{2(N+2)}(I-\tfrac{1}{2}), \quad I = 1, \ldots, N+2. \tag{2.14}$$

For simplicity, we shall consider only values of γ of the form

$$\gamma = \frac{\pi}{2(N+2)}M, \tag{2.15}$$

where M is an integer smaller than $N+2$. We determine the functions $\tilde{u}$ and $\tilde{v}$ and their derivatives $\tilde{u}_\sigma$ and $\tilde{v}_\sigma$ in terms of the $N+3$ unknowns A, B, λ and a_i $(i = 1, \ldots, N)$ by substituting $t = \mathrm{e}^{\mathrm{i}\sigma}$ into (2.11). We find the $N+3$ unknowns by satisfying (2.13) at the mesh points $\sigma_I, I = 1, \ldots, M$ and (2.7) at the mesh points $\sigma_I, I = M+1, \ldots, N+2$. Thus we obtain $N+2$ nonlinear algebraic equations. The last equation is obtained by imposing (2.5). This system of equations is solved by using Newton's method.

3. Numerical results

The numerical scheme described in the previous section was used to compute solutions for various values of F and γ. Typical profiles are shown in figure 1. As $\gamma \to 0$, the length L of the plate tends to infinity and the numerical results approach the semi-infinite plate results of Vanden-Broeck & Keller (1987). As $\gamma \to \frac{1}{2}\pi$ we find that $L \to 0$ and the flow reduces to a solitary wave. Values of L versus F^2 for $\gamma = 5\pi/16$ are shown in figure 2.

We now define the amplitude parameter

$$\alpha = W/H \tag{3.1}$$

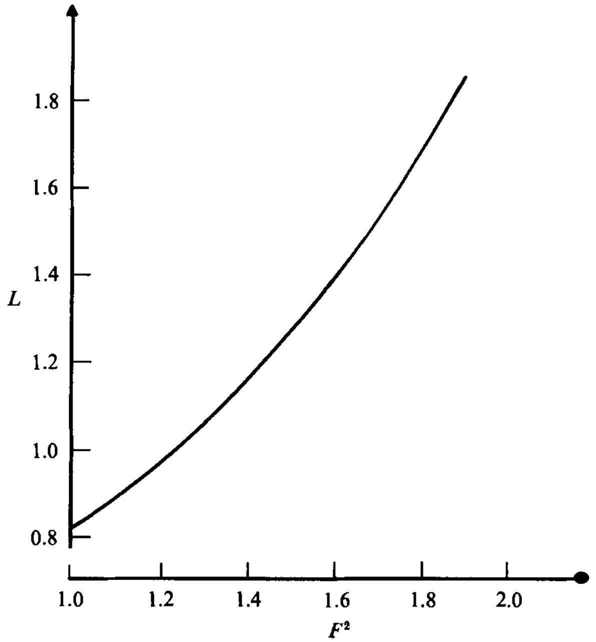

FIGURE 2. The length L of the plate as a function of F^2 for $\gamma = 5\pi/16$.

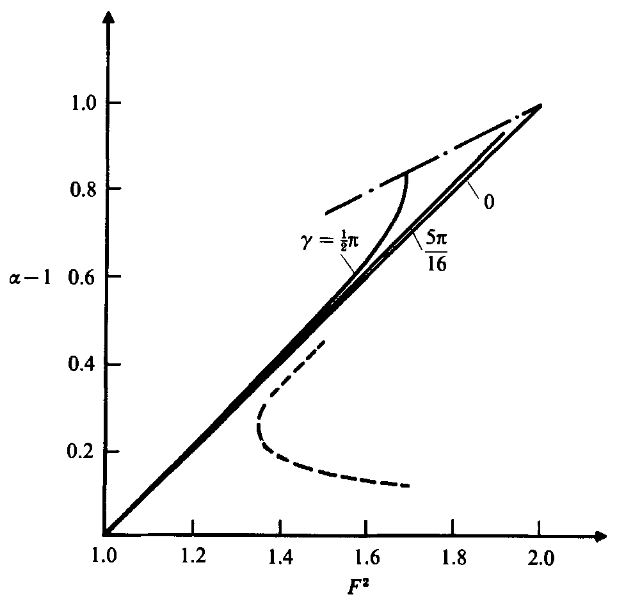

FIGURE 3. The dimensionless height of the board above the free surface level at infinity is $(W-H)/H = \alpha - 1$. It is shown here as a function of F^2 for three values of γ. The lower curve for $\gamma = 0$ corresponds to a semi-infinite plate, the middle curve for $\gamma = 5\pi/16$ corresponds to a plate of the finite length L shown in figure 5, and the upper curve for $\gamma = \frac{1}{2}\pi$ corresponds to a solitary wave with no plate, i.e. $L = 0$. The dot–dash line at the top of the figure is calculated from (3.2), i.e. $\alpha - 1 = \frac{1}{2}F^2$. It corresponds to the limiting cases in which stagnation points with 120° angles occur at the ends of the board. The dashed curve at the bottom of the figure is for flow over a semicircular obstacle of radius $0.2H$ on the bottom of the channel (Vanden-Broeck 1987).

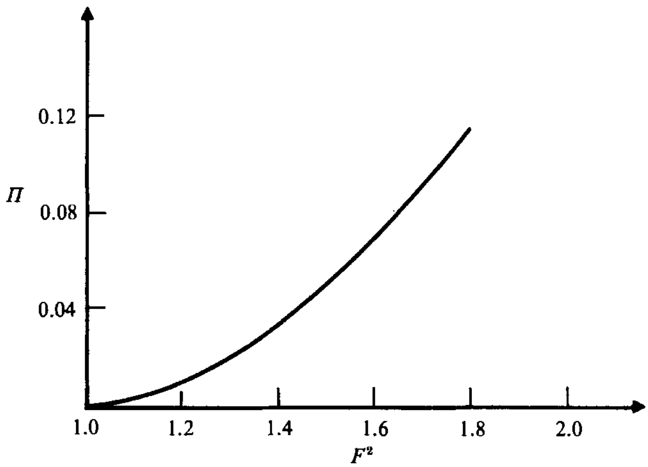

FIGURE 4. The dimensionless vertical force Π exerted on the horizontal board as a function of F^2 for $\gamma = 5\pi/16$.

(see figure 1). In figure 3 we present numerical values of $\alpha - 1$ versus F^2 for various values of γ. For $\gamma = 0$ the results are described by the exact relation $F^2 = \alpha$ derived by Vanden-Broeck & Keller (1987). For $\gamma = \frac{1}{2}\pi$, the numerical values of α as a function of F^2 agree with the solitary-wave results derived by Hunter & Vanden-Broeck (1983) and others.

The dot-dash line in figure 3 corresponds to solutions with stagnation points and 120° angles at the separation points. The equation of this line is obtained by substituting $y = \alpha - 1$ and $q = 0$ into (2.2):

$$\alpha - 1 = \tfrac{1}{2}F^2. \tag{3.2}$$

As $\alpha \rightarrow 1$, each of the two free surfaces approaches half a solitary wave of small amplitude. Thus the equation of the free surfaces as $\alpha \rightarrow 1$ tends to the well-known solution of the Korteweg-de Vries equation:

$$y = (\alpha - 1)\,\mathrm{sech}^2\left[\left(\frac{3}{4(1+\alpha)}\right)^{\frac{1}{2}}(x \pm \tfrac{1}{2}L)\right], \quad |x| > \tfrac{1}{2}L, \tag{3.3}$$

$$F = \alpha^{\frac{1}{2}}. \tag{3.4}$$

Since we neglect viscosity, the only forces acting on the plate are normal pressure forces. By using Bernoulli's equation, we find that the dimensionless force Π acting on the plate is given by

$$\Pi = \frac{1}{\rho U^2 H}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} p\,\mathrm{d}x = \frac{1}{2}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} [1 - q^2 - 2F^{-2}(\alpha - 1)\,\mathrm{d}x. \tag{3.5}$$

Here ρ is the density of the fluid. Numerical values of Π versus F^2 are presented in figure 4 for $\gamma = 5\pi/16$. These values were obtained by substituting the expansion (2.10) into (3.5) and evaluating the integral numerically. Figure 4 and similar results obtained for other values of γ indicate that $\Pi \neq 0$ unless $F = 1$. The physical significance of this result will be explained in the next section.

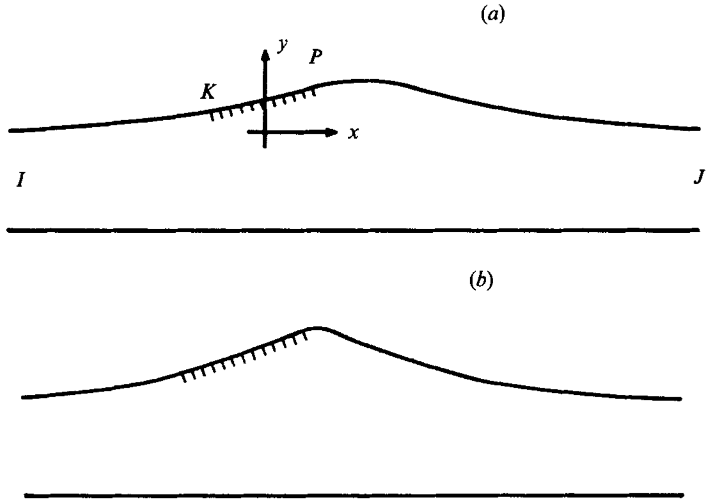

FIGURE 5. Computed profile of flow past a flat surfboard: (*a*) inclined at the angle $\beta = 0.2$ to the horizontal, for $F = 1.225$ and $\gamma = 5\pi/16$; (*b*) inclined at the angle $\beta = 0.3$ to the horizontal with $F = 1.29$ and $\gamma = 5\pi/16$.

4. Flow past an inclined surfboard

We now consider a more realistic surfing flow problem, namely the flow past a flat board or plate inclined at an angle β (see figure 5). The solutions described in §3 are solutions of the present problem for $\beta = 0$.

It is not obvious that there are solutions without a jet for $\beta \neq 0$. Previous calculations (Green 1936; Rispin 1967; Wu 1967; Ting & Keller 1974) show that generally there is a jet or spray thrown up at the leading edge of the plate (see figure 6). (In potential flow the jet must be assumed to disappear into a non-physical second sheet.) We are interested in particular solutions for which the thickness δ of the jet is equal to zero.

To find such solutions, we consider the solutions for $\beta = 0$ considered in §3. Those solutions exist for F^2 in the range $1 \leqslant F^2 \leqslant 2$. For each solution there is a force Π exerted on the board, as is shown in figure 4. For a board inclined at an angle β, if there is no jet there can be no net horizontal force on the board, by conservation of momentum. But then there can be no net normal force either, because the board is flat. Consequently, as $\beta \to 0$ these solutions must approach one of the solutions of the problem with $\beta = 0$ for which $\Pi = 0$. The only such solution is that at $F = 1$. We conclude that if solutions without a jet exist for $\beta \neq 0$, they must branch from the solution with $\beta = 0$ and $F = 1$, which is the uniform horizontal flow.

For a plate that is not flat, zero horizontal force does not imply zero total force. Therefore a surfboard of arbitrary shape will usually have non-zero lift even if it has no drag. Furthermore, the branches of jet-free solutions for such boards correspond to perturbed bifurcation from $F = 1$, not to regular bifurcation, as we shall explain in §6.

For flows with a jet, such as that in figure 6, the horizontal force on the board is $\Pi(\beta)\sin\beta$, where β is the slope angle of the board and $\Pi(\beta)$ is the normal force on it. For β small this dimensionless force is approximately $\Pi(0)\beta$. Therefore the momentum loss in the jet, which is $O(\rho U^2 \delta)$, must equal $\Pi(0)\beta\rho U^2 H$. Thus $\delta/H = O[\beta\Pi(0)]$, and the jet thickness δ is zero only when $\Pi = 0$.

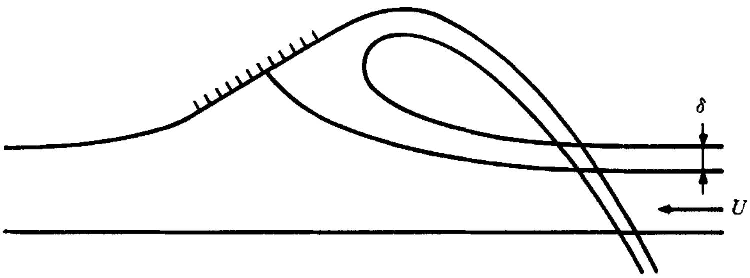

FIGURE 6. Sketch of the flow past an inclined surfboard on the rear face of a wave with a spray or splash jet of thickness δ.

We now generalize the numerical procedure of §2 to compute the flow past a plate inclined at an angle β (see figure 5). In the complex potential plane, $-b$ and $+b$ again denote the values of the potential function at the separation points K and P. As before we map the complex potential plane into the t-plane by using (2.8). The points $t = e^{i\gamma}$ and $t = e^{i(\pi-\gamma)}$ are the images of the separation points K and P in the t-plane. Here γ is defined by (2.9).

We represent the complex velocity ζ by

$$\zeta = e^{\Omega(t)}, \tag{4.1}$$

where $\Omega(t)$ has the expansion

$$\begin{aligned}\Omega(t) = {} & A(1-t)^{2\lambda}+A_1(1+t)^{2\lambda}+B[t^2-2t\cos\gamma+1]^{\frac{1}{2}}+B_1[t^2+2t\cos\gamma+1]^{\frac{1}{2}} \\ & -B[2-2\cos\gamma]^{\frac{1}{2}}-B_1[2+2\cos\gamma]^{\frac{1}{2}}-A_1 2^{2\lambda}+\sum_{n=1}^{\infty} a_n(t^n-1).\end{aligned} \tag{4.2}$$

The expression (4.1) satisfies the kinematic condition (2.7) on IJ if we require all the coefficients to be real. Furthermore, $\zeta(1) = 1$.

For given values of γ and β, the coefficients A, A_1, B, B_1 and a_n and the Froude number F have to be determined to make (4.2) satisfy $\zeta(-1) = 1$, the Bernoulli condition (2.13) and the kinematic condition

$$v = u\tan\beta \tag{4.3}$$

on the plate KP. To achieve this we truncate the infinite series in (4.2) after a finite number of terms and use the collocation procedure described in §2.

The coefficients a_n were found to decrease rapidly. For example,

$$\begin{aligned}&a(1) \sim -0.64, \quad a(10) \sim -0.34\times 10^{-2},\\ &a(40) \sim -0.78\times 10^{-4} \quad \text{for } \gamma = 5\pi/16 \quad \text{and } a = 0.2.\end{aligned}$$

Most of the calculations were performed with 60 coefficients. The results indicate that there is a two-parameter family of solutions. We chose these two parameters to be γ and β.

Typical profiles for $\gamma = 5\pi/16$ with $\beta = 0.2$ and $\beta = 0.3$ are shown in figure 5. In figure 7, we present numerical values of β versus F for $\gamma = 5\pi/16$. As expected, the curve approaches $F = 1$ as $\beta \to 0$. We obtained similar results for other values of γ. As $\gamma \to \frac{1}{2}\pi$, the length of the plate approaches zero and the solution reduces to a solitary wave with a plate of arbitrarily small length at one of its inflexion points.

For F close to 1 we expect the equation $y = \eta(x)$ of the free surface to satisfy the

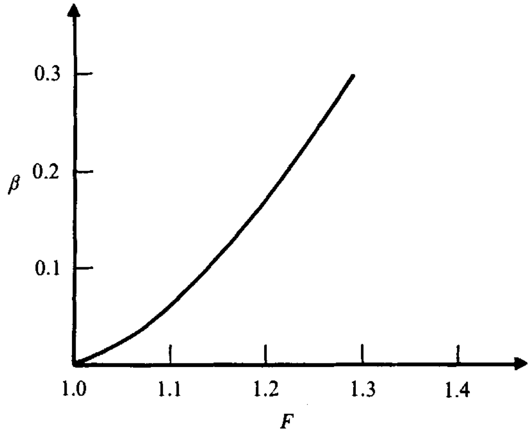

FIGURE 7. The slope angle β is shown as a function of F for $\gamma = 5\pi/16$.

Korteweg–de Vries equation. Integrating this equation and using the fact that $\eta'(x)$ and $\eta(x)$ aporoach zero as $|x| \to \infty$ we obtain

$$[\eta'(x)]^2 = 3[\eta(x)]^2\{1-F^{-2}[1+\eta(x)]\}. \tag{4.4}$$

At the separation points K and P the slope of the free surface should be equal to $\tan\beta$. Thus, we should have

$$[\tan\beta]^2 = 3(y_K)^2[1-F^{-2}(1+y_K)], \tag{4.5}$$

$$[\tan\beta]^2 = 3(y_P)^2[1-F^{-2}(1+y_P)]. \tag{4.6}$$

Here y_K and y_P are the ordinates of the separation points K and P. Since the length of the plate is L we have the additional relation

$$y_\mathrm{P} - y_\mathrm{K} = L\sin\beta. \tag{4.7}$$

For a given value of L, relations (4.5)–(4.7) define F implicitly in terms of β. As a check on our numerical calculations we substitute our numerical values for y_K and y_P into (4.5) and (4.6) for $\beta = 0.1$ and solve for F. We obtain the values $F = 1.151$ and $F = 1.155$. These values agree within 1% with the value $F = 1.142$ obtained from our numerical solution of the flow problem. This indicates that the free surface does indeed approach that of a small-amplitude solitary wave.

Finally we recall that the direction of a potential flow can be reversed. Therefore, the profiles in figure 5 represent solutions for flows from right to left as well as for flows from left to right. Thus the board can be on either the front or rear face of the wave. However, it seems that it is stable only when it is on the front face.

5. Flow past a submerged hydrofoil

We shall now consider the flow past a horizontal flat plate of length L which is submerged beneath the surface of the fluid. Such a submerged plate may be thought of as a two-dimensional hydrofoil. The configuration can be represented by figure 1 with the plate KP submerged to a distance $D < W$ above the bottom, with W still denoting the maximum height of the free surface. For every value of the plate length L, and every value of $D < H$, a uniform horizontal flow is an exact solution of this

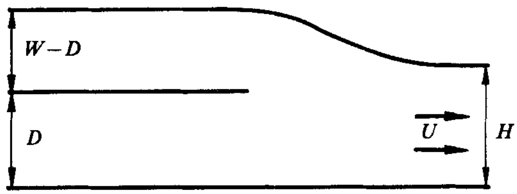

FIGURE 8. Sketch of the flow past a semi-infinite horizontal hydrofoil.

problem. For these flows $\alpha = W/H = 1$, so they all lie on the F^2 axis in figure 3. We expect that a one-parameter family of branches of solutions without singularities at the ends of the plate will bifurcate from the point $F = 1, \alpha - 1 = 0$, just as for the surfing flows shown in figure 3. The branch of solitary waves for which $L = 0$ will be the highest branch, while the branch with $L = \infty$ will be the lowest branch.

When $L = \infty$ the hydrofoil is semi-infinite, as is shown in figure 8. For that case we can use conservation of mass and of the horizontal component of momentum to obtain an exact expression relating α to D/H and F. The calculation is similar to that in Vanden-Broeck & Keller (1987), and the result is

$$\left(\frac{D}{H}\right)^{-1}\left\{1-[1+2F^{-2}(1-\alpha)]^{\frac{1}{2}}\left(\alpha-\frac{D}{H}\right)\right\}^2+[1+2F^{-2}(1-\alpha)]\left(\alpha-\frac{D}{H}\right)$$
$$+\alpha+\frac{2\alpha}{F^2}-\frac{\alpha^2}{F^2}-\frac{1}{F^2}-2=0. \quad (5.1)$$

This solution yields $\alpha - 1$ versus F^2 on the branch of solutions with $L = \infty$.

The dot–dash curve in figure 3, corresponding to a stagnation point at the crest of the free surface, will still be given by (3.2). The vertical force or lift on the hydrofoil will be given by a curve like that in figure 4.

When the plate is inclined at an angle β to the horizontal, we also expect that there will be a two-parameter family of flows without singularities at the ends of the plate. For them the inclination β will vary with F in the same way as is shown in figure 7. The flows will all bifurcate from $F = 1$, $\beta = 0$. There will be no drag or lift on the plate. As the plate is moved toward the free surface, the flow will approach the surfing flow without any jet. Thus the flow around a submerged object will go over smoothly to the flow past an object in the surface without producing a jet.

6. Conclusion

We have constructed solutions without splash or spray jets for the flow past horizontal and inclined flat surfboards. The flows bifurcate from the uniform stream at $F = 1$ (see figures 3 and 7). Since fluid motion without the surfboard is a solitary wave, these solutions can be viewed as perturbed solitary waves. They are perturbed by flat surfboards on their free surfaces.

Our numerical procedure could be generalized to investigate solitary waves perturbed by surfboards that are not flat, or by submerged obstacles. For such perturbations the uniform stream is not a solution for any value of F. Therefore these flows do not bifurcate from the uniform stream at $F = 1$. Instead we may expect perturbed bifurcation from $F = 1$.

This expectation is confirmed by the numerical results of Vanden-Broeck (1987). He investigated solitary waves perturbed by a semicircular obstacle on the bottom of the channel. The dashed line in figure 3 shows his numerical values of $\alpha - 1$ versus F^2 for a semicircle of radius $0.2H$. This curve can be interpreted as a perturbed bifurcation from $F = 1$. Similar results should be expected for non-flat surfboards.

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